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DISTRIBUTION OF POINTS ON SPHERES AND APPROXIMATION BY ZONOTOPES

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ABSTRACT

It is proved that if we approximate the Euclidean ball B^n in the Hausdorff distance up to ε by a Minkowski sum of N segments, then the smallest possible N is equal (up to a possible logarithmic factor) to $c(n)\varepsilon^{-2(n-1)/(n+2)}$. A similar result is proved if B^n is replaced by a general zonoid in $\mathbf{R}^n$.

The topic of approximating convex bodies by zonotopes (i.e., polytopes which are Minkowski sums $\Sigma_{j=1}^N I_j$ of segments I_j) was the subject of several papers in recent years. The main question is how many summands N are needed in order to approximate a given n-dimensional zonoid Z up to ε, in the Hausdorff metric, say (a zonoid is by definition a convex body which can be approximated arbitrarily well by zonotopes). We are interested here in the dependence of N on ε for a fixed dimension n and, more specifically, for the case where Z is the usual Euclidean ball B^n in $\mathbf{R}^n$. It is easy to see and well known that for $Z = B^n$ the question can be formulated equivalently as follows: How many directions $\{y_j\}_{j=1}^N \subset S^{n-1} = \partial B^n$ are needed in order to be able to determine the surface area of a convex body K in $\mathbf{R}^n$ (up to a relative error of ε) by knowing the $(n-1)$-dimensional volumes of the projections of K on the hyperplanes orthogonal to $\{y_j\}_{j=1}^N$. The determination of the $\{y_j\}_{j=1}^N$ themselves is a question of finding a good distribution of points on S^{n-1}.

The specific question for $Z = B^n$ was treated e.g. in [4], [2], [3] and [5]. In [3] it was proved by using spherical harmonics that if $P = \Sigma_{j=1}^N I_j$ is a zonotope with $B^n \subset P \subset (1+\varepsilon)B^n$ then

$$N \geqq c_1(n)\varepsilon^{-2(n-1)/(n+2)}. \tag{1}$$

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Our aim here is to prove that up to a possible logarithmic factor (1) is the right estimate (we assume $n \geqq 3$ since for $n = 2$ the situation is trivial).

THEOREM 1. *For $n \geqq 3$ there is a constant $c_2(n)$ so that for every $0 < \varepsilon < 1/2$ there is a zonotope $P = \Sigma_{j=1}^N I_j$ so that $B^n \subset P \subset (1+\varepsilon)B^n$ and*

$$N \leqq c_2(n)(\varepsilon^{-2}|\log \varepsilon|)^{(n-1)/(n+2)}. \tag{2}$$

PROOF. By straightforward duality the statement of the theorem is equivalent to the following. There are $\{z_j\}_{j=1}^N$ in S^{n-1} and non-negative scalars $\{\alpha_j\}_{j=1}^N$ with N given by (2) so that

$$\left| \sum_{j=1}^{N} \alpha_j |\langle x, z_j\rangle| - \beta_n \right| \leqq \varepsilon, \qquad x \in S^{n-1} \tag{3}$$

where β_n is a constant which we choose to take as $\int_{S^{n-1}} |\langle x, z\rangle| \, d\mu(z)$ where μ denotes the normalized rotation invariant measure on S^{n-1}.

We shall apply an idea which has already proved useful in other questions concerning distribution of points on spheres (cf. [1]; the connection of the problem of approximating B^n by zonotopes to the questions treated in [1] was pointed out in [5]). The surface of the ball is first cut by a deterministic procedure into small pieces. In each of these pieces points are then chosen at random. Here we shall choose randomly in each small piece a configuration of $n+2$ points. The points in each such configuration are strongly dependent but the configurations in the different pieces are chosen independently of each other.

Let N be an integer (for the time being arbitrary; the connection with ε via (2) will enter only at the end of the proof) and put

$$\eta = N^{-1/(n-1)}. \tag{4}$$

We partition S^{n-1} into N compact connected sets $\{Q_j\}_{j=1}^N$ with $\mu(Q_j) = N^{-1}$ for every j (and thus $\mu(Q_{j_1} \cap Q_{j_2}) = 0$ for $j_1 \neq j_2$) and so that the diameter of each Q_j is $\leqq c_3(n)\eta$. For each j let μ_j be the probability measure $N\mu_{|Q_j}$ on Q_j and let Σ_j be the set of probability measures σ on Q_j of the form

$$\sigma = \sum_{i=1}^{n+2} \lambda_i(\sigma)\delta_{y_i(\sigma)}, \qquad \lambda_i(\sigma) \geqq 0, \quad \sum_{i=1}^{n+2} \lambda_i(\sigma) = 1, \quad y_i(\sigma) \in Q_j$$

so that

$$\sum_{i=1}^{n+2} \lambda_i(\sigma) f(y_i(\sigma)) = \int_{Q_j} f(y) d\sigma(y) = \int_{Q_j} f(y) d\mu_j(y) = N \int_{Q_j} f(y) d\mu(y) \tag{5}$$

for every function f which is the restriction of a linear function on $\mathbf{R}^n$ to Q_j.

We claim that μ_j belongs to the w* closed convex hull K_j of Σ_j (in the usual topology on measures, induced by $C(Q_j)$). Indeed, otherwise there would exist a $g \in C(Q_j)$ such that

$$\int_{Q_j} g(y) d\mu_j(y) > \sup \left\{ \int_{Q_j} g(y) d\sigma(y); \sigma \in \Sigma_j \right\}. \tag{6}$$

Consider the map $\pi : Q_j \to \mathbf{R}^{n+1}$ defined by

$$\pi(x) = (x_1, x_2, \ldots, x_n, g(x))$$

where x_k denotes the kth coordinate of $x \in Q_j \subset S^{n-1}$. The point $w = (u_1, \ldots, u_n, \int_{Q_j} g\, d\mu_j)$, where $u = (u_1, \ldots, u_n)$ is the barycenter of μ_j, belongs to Conv $\pi(Q_j)$. Hence, by Carathéodory's theorem, there are $\{y_i\}_{i=1}^{n+2} \in Q_j$ and $\{\lambda_i\}_{i=1}^{n+2}$ with $\lambda_i \geqq 0$, $\Sigma \lambda_i = 1$ so that $w = \Sigma_{i=1}^{n+2} \lambda_i \pi(y_i)$. The measure $\sigma = \Sigma_{i=1}^{n+2} \lambda_i \delta_{y_i}$ belongs to Σ_j and satisfies $\int_{Q_j} g\, d\mu_j = \int_{Q_j} g\, d\sigma$ contradicting (6). Having established that $\mu_j \in K_j$ it follows that there is a probability measure ν_j on the w* compact set Σ_j whose barycenter is μ_j, i.e., for every $f \in C(Q_j)$

$$N \int_{Q_j} f(y) d\mu(y) = \int_{Q_j} f(y) d\mu_j(y) = \int_{\Sigma_j} \sum_{i=1}^{n+2} \lambda_i(\sigma) f(y_i(\sigma)) d\nu_j(\sigma). \tag{7}$$

We choose now on each Q_j a $\sigma \in \Sigma_j$ randomly according to the probability ν_j. The choices for different j are done independently of each other. For each $f \in C(S^{n-1})$ the N independent variables (on $\Pi_{j=1}^N(\Sigma_j, \nu_j)$)

$$h_{j,f}(\sigma) = \sum_{i=1}^{n+2} \lambda_i(\sigma) f(y_i(\sigma)) - \int_{Q_j} f\, d\mu_j$$

all have mean 0 by (7). If f and j are such that $f_{|Q_j}$ is a restriction to Q_j of a linear function, then $h_{j,f} \equiv 0$ by (5). Also, if f satisfies a Lipschitz condition

$$|f(u) - f(v)| \leqq \| u - v \|, \qquad u, v \in S^{n-1} \tag{8}$$

then it follows from our assumption on the diameter of Q_j that for all σ and j

$$|h_{j,f}(\sigma)| \leqq c_3(n)\eta.$$

We recall now a basic fact from probability theory, namely the so-called Bernstein inequality, which in its simplest form reads as follows.

Assume that $\{g_j\}_{j=1}^{J}$ are independent random variables on a probability space with mean 0 and bounded uniformly by 1 in absolute value. Then for $0<\delta<1$

$$\mathrm{Prob}\left\{\left|\sum_{j=1}^{J} g_j\right| > \delta J\right\} \leqq 2\exp(-J\delta^2/2). \tag{9}$$

Let $x \in S^{n-1}$ and consider $f_x(y) = |\langle x, y\rangle| \in C(S^{n-1})$. Clearly f_x satisfies (8). Moreover, the restriction f_x to Q_j is linear provided Q_j does not meet the set $\{y; \langle x, y\rangle = 0\}$. The set of j's for which Q_j meets $\{y; \langle x, y\rangle = 0\}$ has cardinality at most $c_4(n)N\eta = J$. By applying (9) to the family $\{h_{j,f_x}/c_3(n)\eta\}_{j=1}^{N}$ and remembering that only J of them are not identically 0, we get for $0<\delta<1$

$$\begin{aligned} &\mathrm{Prob}\left\{\sigma \in \prod_{j=1}^{N} \Sigma_j;\ \left|\sum_{j=1}^{N} h_{j,f_x}(\sigma_j)\right| > c_3(n)c_4(n)\delta\eta^2 N\right\} \\ &\leqq 2\exp(-c_4(n)\eta N\delta^2/2). \end{aligned} \tag{10}$$

We now recall the value of η from (4) and choose δ so that $c_3(n)c_4(n)\delta\eta^2 = \varepsilon/2$. For N satisfying (2) we have $\delta<1$ for $\varepsilon<\varepsilon_0(n)$. We can now rewrite (10) as follows: The probability of the set $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Pi_{j=1}^{N} \Sigma_j$ so that

$$\left| N^{-1} \sum_{j=1}^{N} \sum_{i=1}^{n+2} \lambda_i(\sigma_j)\,|\langle y_i(\sigma_j), x\rangle| - \int_{S^{n-1}} |\langle y, x\rangle|\, d\mu(y)\right| > \varepsilon/2 \tag{11}$$

is at most

$$2\exp(-N^{(n+2)/(n-1)}\varepsilon^2/c_5(n)).$$

We let now x vary on an $\varepsilon/4$ net in S^{n-1}. The number of points in such a net is $\leqq (c_6(n)/\varepsilon)^{n-1}$. Hence if

$$2(c_6(n)/\varepsilon)^{n-1}\exp(-N^{(n+2)/(n-1)}\varepsilon^2/c_5(n)) < 1$$

(and this is the case for N of the form (2) with suitable $c_2(n)$), then there are $\sigma \in \Pi_{j=1}^{N} \Sigma_j$ for which (11) does not hold for any x in the net. For such σ, the expression in the right-hand side of (11) is at most equal to ε for every $x \in S^{n-1}$ and this is the desired result (3) (with N replaced by $N(n+2)$). □

The argument used to prove Theorem 1 can be modified so as to apply for general zonoids in $\mathbf{R}^n$. For $n=2$ the situation is trivial; any zonoid (which means actually any symmetric convex body in this case) can be approximated up to ε by a zonotope having $c(2)\varepsilon^{-1/2}$ summands. For $n=3$ we obtain for general zonoids the same result as for B^3. For $n=4$ we get for general zonoids a result which is only marginally worse than that for B^4 (a somewhat larger logarithmic factor). For $n \geqq 5$ our proof gives for general zonoids a result with a worse exponent than that for B^n.

THEOREM 2. *For $n \geqq 3$ there is a constant $c(n)$ so that for every $0 < \varepsilon < 1/2$ and every zonoid Z in $\mathbf{R}^n$ (having the origin as an interior point and center) there is a zonotope $P = \Sigma_{j=1}^N I_j$ with $Z \subset P \subset (1+\varepsilon)Z$ and so that*

$$N \leqq c(3)\varepsilon^{-4/5}|\log \varepsilon|^{2/5} \qquad \text{if } n = 3, \tag{12}$$

$$N \leqq c(4)\varepsilon^{-1}|\log \varepsilon|^{3/2} \qquad \text{if } n = 4, \tag{13}$$

$$N \leqq c(n)(\varepsilon^{-2}|\log \varepsilon|)^{(n-2)/n} \qquad \text{if } n \geqq 5. \tag{14}$$

PROOF. The argument in all three cases is very similar. We shall present here only the proof for $n = 4$. The outline of the proof follows that of Theorem 1. We just point out the places where the argument differs.

By passing to the dual space we see that what we have to show is the following. Let τ be a probability measure on S^3, then there exist N points $\{z_j\}_{j=1}^N$ on S^3 and non-negative scalars $\{\alpha_j\}_{j=1}^N$ so that

$$\left| \sum_{j=1}^N \alpha_j |\langle z_j, x\rangle| - \int_{S^3} |\langle z, x\rangle| \, d\tau(z) \right| < \varepsilon, \qquad x \in S^3. \tag{15}$$

There is no loss of generality to assume that τ is absolutely continuous with respect to the rotation invariant measure μ on S^3 and that it has positive density everywhere.

Let N be an integer and put again $\eta = N^{-1/3}$. We partition S^3 into $M \leqq c_1 N$ compact sets $\{Q_j\}_{j=1}^M$ with $\mu(Q_{j_1} \cap Q_{j_2}) = 0$ for $j_1 \neq j_2$ as follows. For every $u \in S^3$ let r_u be the radius of the cap on S^3 with center u whose τ measure is N^{-1}. We let C_u be the cap on S^3 with center u and radius $\min(r_u, \eta)$. The caps $\{C_u\}_{u \in S^3}$ form a covering of S^3. By the Besicovitch covering theorem (cf. [6] Theorem 1.2.1) there is a subcovering $\{C_{u_j}\}_{j=1}^M$ of $\{C_u\}_{u \in S^3}$ which has a bounded multiplicity. (Every point in S^3 is covered at least once and at most c_2 times. The constant c_2 does not depend on N; it depends just on the dimension which now is 3.) Since for each u

$$\max(\tau(C_u), \mu(C_u)) \geqq c_3 N^{-1}$$

it is clear that the M we get is $\leqq c_1 N$ for some c_1. We next replace each C_{u_j} by a compact subset Q_j so as to obtain disjoint sets (up to measure 0). Observe that

$\tau(Q_j) \leqq N^{-1}$ for every j. We assume that $\tau(Q_j) > 0$ for every j (otherwise we simply discard some indices).

On each Q_j we define the set Σ_j and the measure ν_j on Σ_j as in the proof of Theorem 1; we just replace $N^{-1}\mu_{|Q_j}$ there by $\tau(Q_j)^{-1}\tau_{|Q_j}$ in the present context. The random variables to which we apply the Bernstein inequality will now be normalized as follows. For $f \in C(S^3)$ and $1 \leqq j \leqq M$

$$k_{j,f}(\sigma) = \sum_{i=1}^{6} \tau(Q_j)\lambda_i(\sigma) f(y_i(\sigma)) - \int_{Q_j} f(y)d\tau(y). \tag{16}$$

These variables have ν_j mean 0, vanish if $f_{|Q_j}$ is linear, and if f satisfies (8) then

$$| k_{j,f}(\sigma) | \leqq N^{-1} \cdot \text{diameter}(Q_j). \tag{17}$$

Let $x \in S^3$ and put $f_x = |\langle x, y \rangle| \in C(S^3)$. For the argument in the proof of Theorem 1 it is important to know on how many sets Q_j, $f_{x|Q_j}$ fails to be linear and thus, for how many j does C_{u_j} intersect $\{y; \langle x, y \rangle = 0\}$. For $\rho > 0$ put

$$A_\rho = \{j; C_{u_j} \cap \{y; \langle x, y \rangle = 0\} \neq \varnothing, \text{ radius } C_{u_j} \text{ between } \rho \text{ and } 2\rho\}.$$

Let J_ρ be the cardinality of A_ρ. By the fact that the covering $\{C_{u_j}\}_{j=1}^{M}$ has finite multiplicity, we deduce that

$$J_\rho \cdot \rho^3 \leqq c_4 \cdot \rho.$$

Since always $J_\rho \leqq M$ and $\rho \leqq \eta$, we may restrict our attention to the range $[\sim N^{-1/2}, N^{-1/3}]$, and since we allow radii between ρ and 2ρ, only $c_5 \log N$ sets A_ρ appear.

From (9), we get for $0 < \delta < 1$ and some $c_6 > 0$

$$\text{Prob}\left\{\sigma; \left| \sum_{j \in A_\rho} k_{j,f_x}(\sigma) \right| > c_6 \rho J_\rho \delta N^{-1}\right\} \leqq 2 \exp(-J_\rho \delta^2/2). \tag{18}$$

We choose now δ so that $c_6 \rho J_\rho \delta N^{-1} = \varepsilon / c_5 \log N$. Then (18) becomes

$$\text{Prob}\left\{\sigma; \left| \sum_{j \in A_\rho} k_{j,f_x}(\sigma) \right| > \varepsilon / c_5 \log N\right\} \leqq 2 \exp(-N^2\varepsilon^2/c_7 \log^2 N). \tag{19}$$

By summing (19) over all $c_5 \log N$ sets A_ρ we get that

$$\text{Prob}\left\{\sigma; \left| \sum_{j=1}^{M} \sum_{i=1}^{6} \tau(Q_j)\lambda_i(\sigma) f_x(y_i(\sigma)) - \int_{S^3} f_x(y)d\tau(y) \right| > \varepsilon\right\}$$

is at most $2c_5 \log N \exp(-N^2\varepsilon^2/c_7 \log^2 N)$. We conclude the proof as in the case of Theorem 1 by taking nets of vectors x on S^3. We deduce that for N as in (13), the required inequality (15) holds for appropriate α_j and z_j and all x provided N is replaced by $6M$ and ε by 2ε. □

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